# (NON)LOCAL HAMILTONIAN AND SYMPLECTIC STRUCTURES, RECURSIONS, AND HIERARCHIES: A NEW APPROACH AND APPLICATIONS TO THE $N = 1$ SUPERSYMMETRIC KDV EQUATION

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Abstract. Using methods of [7] and [8], we accomplish an extensive study of the $N = 1$ supersymmetric Korteweg-de Vries equation. The results include: a description of local and nonlocal Hamiltonian and symplectic structures, five hierarchies of symmetries, the corresponding hierarchies of conservation laws, recursion operators for symmetries and generating functions of conservation laws. We stress that the main point of the paper is not just the results on super-KdV equation itself, but merely exposition of the efficiency of the geometrical approach and of the computational algorithms based on it.

## Introduction

There exists a number of super extensions of the classical KdV equation

$$u_t = -u_{xxx} + 6uu_x$$

(see [15] and the references therein). One of them, the so-called $N = 1$ supersymmetric extension, is

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x + \varphi_{xx}\varphi, \\ \varphi_t &= -\varphi_{xxx} + 3u\varphi_x + 3u_x\varphi, \end{aligned} \tag{1}$$

where $\varphi$ is an odd (fermionic) variable, [17]. To deal with this system, it is convenient to introduce a new independent odd variable $\theta$ such that $D_\theta^2 = D_x$, where

$$D_\theta = \partial_\theta + \theta D_x$$

(here $D_x$ denotes the total derivative operator; see below) and a new odd function

$$\Phi = \varphi + \theta u.$$

Then (1) will acquire the form

$$\Phi_t = -\Phi_{xxx} + 3D_\theta(\Phi)\Phi_x + 3D_\theta(\Phi_x)\Phi. \tag{2}$$

This equation is linear in $\theta$ and reduces to (1) if we equal to each other the corresponding coefficients at the left- and right-hand sides. System (1) (or equation (2)) was studied before (see, e.g., [13]) and a number of results related to its integrability were obtained. The aim of our paper is twofold: (1) to represent the known results in a more convenient form (at least, from our point of view); (2) to demonstrate the efficiency of new methods of analysis of integrable systems described in [7, 8] and based on a general geometric approach to nonlinear PDE [2, 10]. Actually, description of *these methods and their highly algorithmical nature* (and, to a less extent, of the results on the super-KdV equation themselves) is the main goal of the paper.

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For traditional approach to Hamiltonian formalism in integrable systems we refer the reader to [3, 12, 19]; an extensive exposition of the theory for superintegrable systems can be found in [11].

This paper is organized as follows. In Section 1, we present the essential definitions and results needed for applications paying main attention to the computational aspects rather than to theoretical ones. All the proofs can be found in [2, 7, 8, 10]. In Section 2, the results for the $N = 1$ supersymmetric KdV equation are described. Finally, in the last section we briefly discuss the results and perspectives.

## 1. Description of the computational scheme

Here we deal with evolution systems $\mathscr{E}$ of the form

$$v_t = F(y, t, v_1, \dots, v_k), \tag{3}$$

where both the unknown variable $v = (v^1, \dots, v^m)$ and the right-hand side $F = (F^1, \dots, F^m)$ are vector-functions and $v_i = \partial^i v/\partial y^i$, $y$ and $t$ being the independent variables.

*Remark* 1. In applications, some of the variables $v^j$, as well as $y$, may be *odd*. In particular, in equation (2) $\theta$ and $\Phi$ are odd and $x$ is even. Nevertheless, for the sake of simplicity, we expose the general theory for purely *even* equations. Necessary corrections needed for the super case the reader will find in Subsection 1.10.

Two basic operators related to (3),

$$D_y = \frac{\partial}{\partial y} + \sum_{i,j} v^j_{i+1} \frac{\partial}{\partial v^j_i}, \quad D_t = \frac{\partial}{\partial t} + \sum_{i,j} D^i_y(F^j) \frac{\partial}{\partial v^j_i},$$

are called the *total derivatives.*

*Remark* 2. Note that the above expressions for total derivatives contain infinite number of terms. To make the action of these operators (as well as of similar operators introduced below) well defined, we introduce the space $\mathscr{F}(\mathscr{E})$ of functions smoothly depending on $y$, $t$ and a *finite number* of variables $v^j_i$, and assume $D_y$ and $D_t$ to act in this space. Similarly, we shall consider the spaces $\mathscr{F}^m(\mathscr{E})$ of vector-functions of length $m$ that depend on $y$, $t$ and $v^j_i$ in the same way.

1.1. **Symmetries.** A *symmetry* of equation (3) is a vector field

$$S = \sum_{i,j} S^j_i \frac{\partial}{\partial v^j_i}, \quad S^j_i \in \mathscr{F}(\mathscr{E}),$$

such that

$$[S, D_y] = [S, D_t] = 0.$$

Any symmetry is of the form

$$\ni_f = \sum_{i,j} D^i_y(f^j) \frac{\partial}{\partial v^j_i}, \tag{4}$$

where the vector-function $f = (f^1, \dots, f^m) \in \mathscr{F}^m(\mathscr{E})$ satisfies the system of equations

$$D_t(f^l) = \sum_{i,j} \frac{\partial F^l}{\partial v^j_i} D^i_y(f^j), \quad l = 1, \dots, m. \tag{5}$$

The operator at the right-hand side of (5) is called the *linearization* of $F$ and is denoted by $\ell_F$. Thus, equation (5) acquires the form

$$D_t(f) = \ell_F(f). \tag{6}$$

There exists a one-to-one correspondence between symmetries (4) and the corresponding functions $f \in \mathscr{F}^m(\mathscr{E})$, hence we shall identify symmetries with such functions and use the term 'symmetry' for any function that satisfy (6).

1.2. **Conservation laws and generating functions.** A *conservation law* of system (3) is a pair $\Omega = (Y, T)$, $Y$, $T \in \mathscr{F}(\mathscr{E})$, such that

$$D_t(Y) = D_y(T). \tag{7}$$

The function $Y$ is called the *density* of $\Omega$. A conservation law is called *trivial* if $Y = D_y(P)$, $T = D_t(P)$ for some function $P \in \mathscr{F}(\mathscr{E})$.

To any conservation law there corresponds its *generating function* defined by

$$g_\Omega = \frac{\delta Y}{\delta v} = \left(\frac{\delta Y}{\delta v^1}, \dots, \frac{\delta Y}{\delta v^m}\right),$$

where

$$\frac{\delta}{\delta v^j} = \sum_{i \geq 0} (-D_y)^i \circ \frac{\partial}{\partial v_i^j}$$

is the *variational derivative* with respect to $v^j$. Generating functions of conservation laws satisfy the system of equations

$$D_t(g) = -\ell_F^*(g), \tag{8}$$

or

$$D_t(g^l) = -\sum_{i,j} (-D_y)^i \left(\frac{\partial F^j}{\partial v_i^l} g^j\right), \quad l = 1, \dots, m, \tag{9}$$

where $\ell_F^*$ is *adjoint* to the operator $\ell_F$.

Any conservation law is uniquely determined by its generating function and, in particular, $\Omega$ is trivial if and only if $g_\Omega = 0$. Stress that equation (9) may possess solutions that do not correspond to any conservation law of (3).

*Remark* 3. Generating functions are also called *cosymmetries* [1] or *conserved covariants* [4].

1.3. **Nonlocal variables.** Let us introduce a set of variables $w^1, \dots, w^j, \dots$ satisfying the equations

$$w_y^j = A^j(y, t, \dots, v_i^\alpha, \dots, w^\beta, \dots), \quad w_t^j = B^j(y, t, \dots, v_i^\alpha, \dots, w^\beta, \dots), \tag{10}$$

that are compatible modulo equation (3), where $A^j$, $B^j$ are some smooth functions depending on a finite number of arguments. Consider the operators

$$\widetilde{D}_y = D_y + \sum_j A^j \frac{\partial}{\partial w^j}, \quad \widetilde{D}_t = D_t + \sum_j B^j \frac{\partial}{\partial w^j}.$$

Due to the compatibility conditions, one has

$$[\widetilde{D}_y, \widetilde{D}_t] = 0 \tag{11}$$

modulo (3). The variables $w^j$ are called *nonlocal*.

Using the operators $\widetilde{D}_y$, $\widetilde{D}_t$ instead of $D_y$ and $D_t$ in formulas (5), (7), and (9), we can introduce the notions of *nonlocal symmetries*, *nonlocal conservation laws*, and *nonlocal generating functions* depending on the new variables $w^j$. We shall denote the spaces of such symmetries and generating functions by $\mathbf{sym}(\mathscr{E})$ and $\mathbf{gf}(\mathscr{E})$, respectively.

*Remark* 4. An invariant geometric way to introduce nonlocal variables is based on the notion of *covering*, see [2, 8, 9, 10].

1.4. **The $\ell$- and $\ell^*$-extensions.** There are two canonical ways to extend the initial system (3). The first one is related to the operator $\ell_F$ and is called the *$\ell$-extension*. Namely, let us introduce the nonlocal variables $\omega_i^j$ (we shall also denote $\omega_0^j$ by $\omega^j$), $j = 1, \dots, m$, $i = 0, 1, \dots$, satisfying the relations

$$(\omega_i^j)_y = \omega_{i+1}^j, \quad (\omega_i^j)_t = \widetilde{D}_y^i\Big(\sum_{s,l} \frac{\partial F^j}{\partial v_s^l}\omega_s^l\Big).$$

Clearly, these equations are consistent modulo (3) and are the consequences of the following ones

$$\omega_t^j = \sum_{i,l} \frac{\partial F^j}{\partial v_i^l}\omega_i^l. \tag{12}$$

In a similar way we construct the *$\ell^*$-extension*: the nonlocal variables are $p_i^j$ ($p_0^j$ will also be denoted by $p^j$) and the defining relations are

$$(p_i^j)_y = p_{i+1}^j, \quad (p_i^j)_t = -\widetilde{D}_y^i\Big(\sum_{s,l} (-\widetilde{D}_y)^s\Big(\frac{\partial F^l}{\partial v_s^j}p^l\Big)\Big),$$

that reduce to the equations

$$p_t^j = -\sum_{s,l} (-\widetilde{D}_y)^s\Big(\frac{\partial F^l}{\partial v_s^j}p^l\Big) \tag{13}$$

and their differential consequences.

*Remark* 5. The parities of the variables $\omega^j$ and $p^j$ are opposite to that of $v^j$: if $v^j$ is *even*, then $\omega^j$ and $p^j$ are *odd* and vice versa.

If the initial equation $\mathscr{E}$ was extended by nonlocal variables $w^j$, we can associate to these variables, in a canonical way, the corresponding $\omega$'s and $p$'s whose 'behavior' is governed by linearization or, respectively, adjoint linearization of equations (10) in the corresponding nonlocal setting.

*Associating operators to functions on the $\ell$- and $\ell^*$-extensions.* Let $\mathscr{F}^m(\mathscr{E})$ be the space of vector-valued functions of length $m$ (see Remark 2). Consider the case when $\mathscr{E}$ is not extended by nonlocal variables first. Let $a = (a_1, \dots, a_m)$, $a_i = \sum_{jl} a_l^{ij}\omega_l^j$, $a_l^{ij} \in \mathscr{F}(\mathscr{E})$, be a linear in $\omega$ vector-function. Then we put into correspondence to this function a differential operator $\Delta_a = \|\Delta_a^{ij}\|\colon \mathscr{F}^m(\mathscr{E}) \to \mathscr{F}^m(\mathscr{E})$, where

$$\Delta_a^{ij} = \sum_l a_l^{ij} D_y^l, \qquad i, j = 1, \dots, m.$$

If $\mathscr{F}(\mathscr{E})$ contains nonlocal variables, the situation becomes more complicated. We shall consider here the simplest case when the functions $A^j$ in (10) are independent of $\omega^\beta$. Let $\bar{\omega}^\beta$ be the variable in the $\ell$-extension associated to the nonlocal variable $w^\beta$ and $b = (b^1, \dots, b^m)$, $b^i = \sum_\beta b^{i\beta}\bar{\omega}^\beta$, be a linear in $\bar{\omega}$ vector-function. Then the corresponding operator $\Delta_b = \|\Delta_b^{ij}\|\colon \mathscr{F}^m(\mathscr{E}) \to \mathscr{F}^m(\mathscr{E})$ is of the form

$$\Delta_b^{ij} = \sum_\alpha b^{i\alpha} D_y^{-1} \circ \sum_l \frac{\partial A^\alpha}{\partial v_l^j} D_y^l. \tag{14}$$

For the $\ell^*$-extension the construction is completely similar.

Below we shall use the notation $\mathscr{L}^m(\ell_{\mathscr{E}})$ and $\mathscr{L}^m(\ell_{\mathscr{E}}^*)$ for the spaces of vector-functions linear in $\omega$, $\bar{\omega}$ and $p$, $\bar{p}$, respectively.

1.5. **Recursion operators for symmetries.** Let $R \in \mathscr{L}^m(\ell_{\mathscr{E}})$ be a function that satisfies the equation

$$\widetilde{D}_t(R) = \widetilde{\ell}_F(R).$$

Then the corresponding operator $\Delta_R$ maps $\mathbf{sym}(\mathscr{E})$ to $\mathbf{sym}(\mathscr{E})$ and thus is a recursion operator for (nonlocal) symmetries of $\mathscr{E}$.

*Remark* 6. Here and below by $\widetilde{\ell}_F$ (or $\widetilde{\ell}_F$) we denote the linearization operator (or the adjoint one) in which the total derivatives $D_y$ are substituted by the operators $\widetilde{D}_y$ in the $\ell$- or $\ell^*$-extensions.

1.6. **Recursion operators for generating functions.** Let $L \in \mathscr{L}^m(\ell^*_{\mathscr{E}})$ be a function that satisfies the equation

$$\widetilde{D}_t(L) = -\widetilde{\ell}^*_F(L).$$

Then the corresponding operator $\Delta_L$ maps $\mathbf{gf}(\mathscr{E})$ to $\mathbf{gf}(\mathscr{E})$ and thus is a recursion operator for (nonlocal) generating functions of $\mathscr{E}$ (or *adjoint recursion operator* [1]).

1.7. **Hamiltonian structures.** Let $K \in \mathscr{L}^m(\ell^*_{\mathscr{E}})$ be a function that satisfies the equation

$$\widetilde{D}_t(K) = \widetilde{\ell}_F(K).$$

Then the corresponding operator $\Delta_K$ maps $\mathbf{gf}(\mathscr{E})$ to $\mathbf{sym}(\mathscr{E})$. We call such maps *pre-Hamiltonian structures* (they are also known as *Noether operators* [4]). In order $\Delta_K$ to be a true *Hamiltonian structure*, it has to satisfy two conditions: skew-symmetry ($\Delta^*_K = -\Delta_K$) and the Jacobi identity for the corresponding Poisson bracket (that amounts to $[\![\Delta_K, \Delta_K]\!] = 0$, where $[\![\cdot,\cdot]\!]$ is the *variational Schouten bracket*, see [5, 7]). Both these conditions are easily checked in terms of the function $K$.

Namely, if $K = \big\| \sum_{jl} a^{ij}_l p^j_l \big\|$ then we consider the function $W_K = \sum_{ijl} a^{ij}_l p^j_l p^i$ and in terms of $W_K$ the first condition reads

$$\sum_i \frac{\delta W_K}{\delta p^i} p^i = -2W_K, \tag{15}$$

while the second one is

$$\left(\frac{\delta}{\delta v}, \frac{\delta}{\delta p}\right) \sum_i \left(\frac{\delta W_K}{\delta v^i} \frac{\delta W_K}{\delta p^i}\right) = 0, \tag{16}$$

$(\delta/\delta v, \delta/\delta p) = (\delta/\delta v^1, \dots, \delta/\delta v^m, \delta/\delta p^1, \dots, \delta/\delta p^m)$. Note also that the compatibility condition for two Hamiltonian structures $K$ and $K'$ amounts to

$$\left(\frac{\delta}{\delta v}, \frac{\delta}{\delta p}\right) \sum_i \left(\frac{\delta W_K}{\delta v^i} \frac{\delta W_{K'}}{\delta p^i} + \frac{\delta W_{K'}}{\delta v^i} \frac{\delta W_K}{\delta p^i}\right) = 0. \tag{17}$$

The equation $\mathscr{E}$ itself is in the Hamiltonian form if it possesses a Hamiltonian structure $K$ and may be presented as

$$v_t = \Delta_K \frac{\delta Y}{\delta v} \tag{18}$$

for some $Y = (Y^1, \dots, Y^m)$.

1.8. **Symplectic structures.** Let $J \in \mathscr{L}^m(\ell_{\mathscr{E}})$ be a function that satisfies the equation

$$\widetilde{D}_t(J) = -\widetilde{\ell}_F^*(J).$$

Then the corresponding operator $\Delta_J$ maps $\mathbf{sym}(\mathscr{E})$ to $\mathbf{gf}(\mathscr{E})$ and may be called a *presymplectic structure* on $\mathscr{E}$ (alternatively, *inverse Noether operator* [4]). A presymplectic structure is called *symplectic* if it enjoys in addition the following properties. Let $J = \big\| \sum_{jl} b_l^{ij} \omega_l^j \big\|$. Similar to Subsection 1.7, we consider the function $W_J = \sum_{ijl} b_l^{ij} \omega_l^j \omega^i$ and impose the conditions

$$\sum_i \frac{\delta W_J}{\delta \omega^i} \omega^i = -2W_J, \tag{19}$$

i.e., the operator $\Delta_J$ is skew-adjoint, and

$$\left(\frac{\delta}{\delta v}, \frac{\delta}{\delta \omega}\right) \sum_i \frac{\delta W_J}{\delta v^i} \omega^i = 0 \tag{20}$$

that means that the 'form' $W_J$ s is closed. Thus, in our context the term 'symplectic structure' means the same as in classical mechanics, cf. [14].

1.9. **Canonical representation.** As it will be seen below, all the operators constructed in our study are presented in the form

$$\sum_{\alpha \geq 0} c_{ij}^\alpha D_y^\alpha + \sum_\beta d_j^\beta D_y^{-1} \circ e_i^\beta,$$

where $\|c_{ij}^\alpha\|$ is an $m \times m$-matrix, $\|d_j^\beta\|$ is an $m \times l$-matrix, and $\|e_i^\beta\|$ is an $l \times m$-matrix for some $l > 0$ (matrix-valued functions, to be more precise). In the table it is shown how the matrices $d$ and $e$ look for different types of operators.

| Type of operator | Lines of matrix $d$ | Columns of matrix $e$ |
|---|---|---|
| Recursions for symmetries | Symmetry | Generating function |
| Recursions for generating funct. | Generating function | Symmetry |
| Hamiltonian structures | Symmetry | Symmetry |
| Symplectic structures | Generating function | Generating function |

1.10. **Super case.** We shall now assume that all objects under consideration belong to the super setting, i.e., may be either even or odd, which means that they obey the rule

$$AB = (-1)^{AB} BA.$$

Here and below, symbols used at the exponents of $(-1)$ stand for the corresponding parity. Generalization of the above exposed theory to the super case is carried out along the lines of [18, 10].

Then the basic formulas to be used in the calculus described above are:

(1) for evolutionary derivations

$$\partial_\varphi = \sum_{ij} (-1)^{\varphi v_i^j} D_y^i(\varphi^j) \frac{\partial}{\partial v_i^j}$$

(naturally, the parity of $v_i^j$ equals that of $v^j$ plus parity of $y$ times $i$);

(2) for the linearization one has $\ell_f(\varphi) = (-1)^{f\varphi} \partial_\varphi(f)$ that amounts to

$$(\ell_f)_\alpha^\beta = \sum_i (-1)^{(f^\alpha + 1) v_i^\beta} \frac{\partial f^\alpha}{\partial v_i^\beta} D_y^i;$$

(3) for the operator adjoint to $\Delta = \sum_i a_i D_y^i$ one has

$$\Delta^* = \sum_i (-1)^{i + ia_i y + \frac{i(i-1)}{2} y} D_y^i \circ a_i.$$

## 2. Main results for the $N = 1$ supersymmetric KdV equation

Here we apply the theory described above to equation (2)

$$\Phi_t = -\Phi_{xxx} + 3D_\theta(\Phi)\Phi_x + 3D_\theta(\Phi_x)\Phi.$$

We use the notation

$$\Phi_k \quad \text{for} \quad \frac{\partial^{2k}\Phi}{\partial\theta^{2k}} = \frac{\partial^k \Phi}{\partial x^k}$$

and

$$\Phi_{k\frac{1}{2}} \quad \text{for} \quad D_\theta^{2k+1}(\Phi) = D_\theta\left(\frac{\partial^k \Phi}{\partial x^k}\right).$$

The functions $\Phi_k$ are *odd* while $\Phi_{k\frac{1}{2}}$ are *even*, the function $\Phi = \Phi_0$ itself being odd.

*Gradings.* We assign the following gradings $[\cdot]$ to the variables on our equation:

$$[\theta] = -1/2, \quad [x] = -1, \quad [t] = -3, \quad [\Phi] = 3/2.$$

Respectively, we have

$$[\Phi_k] = (2k+3)/2, \quad [\Phi_{k\frac{1}{2}}] = k + 2.$$

With these gradings, equation (2) becomes homogeneous (of grading 9/2) and all constructions below can be considered to be homogeneous as well.

2.1. **Nonlocal functions.** Here we extend the equation $\mathscr{E}$ by four groups of nonlocal variables. We present here their $\theta$-components only; the $x$- and $t$-components are given in [6] (they are found from compatibility conditions (11)).

2.1.1. *Group 1.* This group includes the even variables $q_1$, $q_3$, $q_5$, defined by

$$\begin{aligned}
(q_1)_\theta &= \Phi_0, \\
(q_3)_\theta &= \Phi_0 \Phi_{\frac{1}{2}}, \\
(q_5)_\theta &= \Phi_{\frac{1}{2}}(-\Phi_2 + 2\Phi_0\Phi_{\frac{1}{2}})/2.
\end{aligned}$$

*Gradings*: $[q_1] = 1$, $[q_3] = 3$, $[q_5] = 5$.

2.1.2. *Group 2.* This group includes the odd variables $Q_{\frac{1}{2}}$, $Q_{\frac{5}{2}}$, $Q_{\frac{9}{2}}$ defined by

$$\begin{aligned}
(Q_{\frac{1}{2}})_\theta &= q_1, \\
(Q_{\frac{5}{2}})_\theta &= q_1^3 - 6q_3, \\
(Q_{\frac{9}{2}})_\theta &= -60\Phi_0\Phi_1 q_1 + q_1^5 - 60q_1^2 q_3 + 240q_5.
\end{aligned}$$

*Gradings*: $[Q_{\frac{1}{2}}] = 1/2$, $[Q_{\frac{5}{2}}] = 5/2$, $[Q_{\frac{9}{2}}] = 9/2$.

2.1.3. *Group 3.* This group includes the odd variables $Q_{\frac{3}{2}}$, $Q_{\frac{7}{2}}$, $Q_{\frac{11}{2}}$ defined by

$$\begin{aligned}
(Q_{\frac{3}{2}})_\theta &= \Phi_0 Q_{\frac{1}{2}}, \\
(Q_{\frac{7}{2}})_\theta &= (12\Phi_2 Q_{\frac{1}{2}} + 18\Phi_1 Q_{\frac{1}{2}} q_1 + \Phi_0 Q_{\frac{5}{2}})/3, \\
(Q_{\frac{11}{2}})_\theta &= (360\Phi_4 Q_{\frac{1}{2}} + 5280\Phi_3 Q_{\frac{1}{2}} q_1 - 760\Phi_2 Q_{\frac{5}{2}} + 4680\Phi_2 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 1200\Phi_2 Q_{\frac{1}{2}} q_1^2 \\
&\quad + 60\Phi_1 Q_{\frac{5}{2}} q_1 + \Phi_0 Q_{\frac{9}{2}})/60.
\end{aligned}$$

*Gradings*: $[Q_{\frac{3}{2}}] = 3/2$, $[Q_{\frac{7}{2}}] = 7/2$, $[Q_{\frac{11}{2}}] = 11/2$.

2.1.4. *Group 4.* This group includes the even variables $\bar{q}_1$, $\bar{q}_3$, $\bar{q}_5$ defined by

$$
\begin{aligned}
&(\bar{q}_1)_\theta = Q_{\frac{3}{2}},\\
&(\bar{q}_3)_\theta = -(Q_{\frac{7}{2}} + Q_{\frac{3}{2}}q_1^2),\\
&(\bar{q}_5)_\theta = (12Q_{\frac{11}{2}} + 42Q_{\frac{7}{2}}\Phi_{\frac{1}{2}} + 6Q_{\frac{7}{2}}q_1^2 + 12Q_{\frac{3}{2}}\Phi_{1\frac{1}{2}}q_1 + Q_{\frac{3}{2}}q_1^4 - 24Q_{\frac{3}{2}}q_1q_3)/3.
\end{aligned}
$$

*Gradings*: $[\bar{q}_1] = 1$, $[\bar{q}_3] = 3$, $[\bar{q}_5] = 5$.

*Remark* 7. The last three variables are not used directly in the subsequent computations, but clarify the nonlocal picture and enter in the expressions for the higher terms of hierarchies of symmetries and generating functions.

2.2. **Seed symmetries.** Solving equation (5), which in our case is of the form

$$\widetilde{D}_t(f) = -\widetilde{D}_\theta^6(f) + 3\widetilde{D}_\theta(f)\Phi_1 + 3\Phi_{\frac{1}{2}}\widetilde{D}_\theta^2(f) + 3\widetilde{D}_\theta^3(f)\Phi + 3\Phi_{1\frac{1}{2}}f,$$

where $\widetilde{D}_\theta = \partial_\theta + \theta\widetilde{D}_x$, while $\widetilde{D}_x$ and $\widetilde{D}_t$ are the total derivative operators extended to the nonlocal setting (see Subsection 2.1), we found a number of solutions that serve as seed symmetries for constructing infinite hierarchies and are used to construct *nonlocal vectors* (see Subsection 2.4 below).

These symmetries are:

*The $Y_k$ series.*

$$
\begin{aligned}
Y_1 &= \Phi_1,\\
Y_3 &= \Phi_3 - 3\Phi_1\Phi_{\frac{1}{2}} - 3\Phi_0\Phi_{1\frac{1}{2}},\\
Y_5 &= \Phi_5 - 5\Phi_3\Phi_{\frac{1}{2}} - 10\Phi_2\Phi_{1\frac{1}{2}} + 10\Phi_1\Phi_{\frac{1}{2}}^2 - 10\Phi_1\Phi_{2\frac{1}{2}} + 20\Phi_0\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}}\\
&\quad - 5\Phi_0\Phi_{3\frac{1}{2}}.
\end{aligned}
$$

*The $Y_{k\frac{1}{2}}$ series.*

$$
\begin{aligned}
Y_{\frac{3}{2}} &= -2\Phi_1Q_{\frac{1}{2}} - \Phi_{\frac{1}{2}}q_1 + \Phi_{1\frac{1}{2}},\\
Y_{\frac{7}{2}} &= -12\Phi_3Q_{\frac{1}{2}} - 2\Phi_1Q_{\frac{5}{2}} + 36\Phi_1Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 36\Phi_0Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} + 12\Phi_0\Phi_2\\
&\quad - 6\Phi_0\Phi_1q_1 + 12\Phi_{\frac{1}{2}}^2q_1 - 36\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} - \Phi_{\frac{1}{2}}q_1^3 + 6\Phi_{\frac{1}{2}}q_3 + 3\Phi_{1\frac{1}{2}}q_1^2 - 6\Phi_{2\frac{1}{2}}q_1\\
&\quad + 6\Phi_{3\frac{1}{2}},\\
Y_{\frac{11}{2}} &= 240\Phi_5Q_{\frac{1}{2}} + 40\Phi_3Q_{\frac{5}{2}} - 1200\Phi_3Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} - 2400\Phi_2Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}}\\
&\quad + 2\Phi_1Q_{\frac{9}{2}} - 120\Phi_1Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} + 2400\Phi_1Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 - 2400\Phi_1Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} - 600\Phi_1\Phi_3\\
&\quad + 240\Phi_1\Phi_2q_1 - 120\Phi_0Q_{\frac{5}{2}}\Phi_{1\frac{1}{2}} + 4800\Phi_0Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}} - 1200\Phi_0Q_{\frac{1}{2}}\Phi_{3\frac{1}{2}}\\
&\quad - 480\Phi_0\Phi_4 + 360\Phi_0\Phi_3q_1 + 1920\Phi_0\Phi_2\Phi_{\frac{1}{2}} - 120\Phi_0\Phi_2q_1^2 - 720\Phi_0\Phi_1\Phi_{\frac{1}{2}}q_1\\
&\quad + 1680\Phi_0\Phi_1\Phi_{1\frac{1}{2}} + 20\Phi_0\Phi_1q_1^3 - 120\Phi_0\Phi_1q_3 + 660\Phi_{\frac{1}{2}}^3q_1 - 3540\Phi_{\frac{1}{2}}^2\Phi_{1\frac{1}{2}}\\
&\quad - 40\Phi_{\frac{1}{2}}^2q_1^3 + 240\Phi_{\frac{1}{2}}^2q_3 + 360\Phi_{\frac{1}{2}}\Phi_{1\frac{1}{2}}q_1^2 - 960\Phi_{\frac{1}{2}}\Phi_{2\frac{1}{2}}q_1 + 1200\Phi_{\frac{1}{2}}\Phi_{3\frac{1}{2}}\\
&\quad + \Phi_{\frac{1}{2}}q_1^5 - 60\Phi_{\frac{1}{2}}q_1^2q_3 + 240\Phi_{\frac{1}{2}}q_5 - 720\Phi_{1\frac{1}{2}}^2q_1 + 2400\Phi_{1\frac{1}{2}}\Phi_{2\frac{1}{2}} - 5\Phi_{1\frac{1}{2}}q_1^4\\
&\quad + 120\Phi_{1\frac{1}{2}}q_1q_3 + 20\Phi_{2\frac{1}{2}}q_1^3 - 120\Phi_{2\frac{1}{2}}q_3 - 60\Phi_{3\frac{1}{2}}q_1^2 + 120\Phi_{4\frac{1}{2}}q_1 - 120\Phi_{5\frac{1}{2}}.
\end{aligned}
$$

*The $Z_k$ series.*

$$
\begin{aligned}
Z_1 &= Q_{\frac12}\Phi_{\frac12} + \theta(-2\Phi_1 Q_{\frac12} - \Phi_{\frac12} q_1 + \Phi_{1\frac12}),\\
Z_3 &= (3Q_{\frac32}\Phi_{\frac12}q_1 - 3Q_{\frac32}\Phi_{1\frac12} + Q_{\frac52}\Phi_{\frac12} - 12Q_{\frac12}\Phi_{\frac12}^2 - 3Q_{\frac12}\Phi_{1\frac12}q_1 + 6Q_{\frac12}\Phi_{2\frac12}\\
&\quad + 6\Phi_1 Q_{\frac12}Q_{\frac32} + 6\Phi_0\Phi_1 Q_{\frac12} + \theta(-12\Phi_3 Q_{\frac12} - 2\Phi_1 Q_{\frac52} + 36\Phi_1 Q_{\frac12}\Phi_{\frac12}\\
&\quad + 36\Phi_0 Q_{\frac12}\Phi_{1\frac12} + 12\Phi_0\Phi_2 - 6\Phi_0\Phi_1 q_1 + 12\Phi_{\frac12}^2 q_1 - 36\Phi_{\frac12}\Phi_{1\frac12}\\
&\quad - \Phi_{\frac12}q_1^3 + 6\Phi_{\frac12}q_3 + 3\Phi_{1\frac12}q_1^2 - 6\Phi_{2\frac12}q_1 + 6\Phi_{3\frac12})/3,\\
Z_5 &= (-15Q_{\frac72}\Phi_{\frac12}q_1 + 15Q_{\frac72}\Phi_{1\frac12} + 120Q_{\frac32}\Phi_{\frac12}^2 q_1 - 360Q_{\frac32}\Phi_{\frac12}\Phi_{1\frac12}\\
&\quad - 10Q_{\frac32}\Phi_{\frac12}q_1^3 + 60Q_{\frac32}\Phi_{\frac12}q_3 + 30Q_{\frac32}\Phi_{1\frac12}q_1^2 - 60Q_{\frac32}\Phi_{2\frac12}q_1 + 60Q_{\frac32}\Phi_{3\frac12}\\
&\quad - Q_{\frac92}\Phi_{\frac12} + 40Q_{\frac52}\Phi_{\frac12}^2 - 5Q_{\frac52}\Phi_{\frac12}q_1^2 + 15Q_{\frac52}\Phi_{1\frac12}q_1 - 20Q_{\frac52}\Phi_{2\frac12} - 660Q_{\frac12}\Phi_{\frac12}^3\\
&\quad + 90Q_{\frac12}\Phi_{\frac12}^2 q_1^2 - 390Q_{\frac12}\Phi_{\frac12}\Phi_{1\frac12}q_1 + 960Q_{\frac12}\Phi_{\frac12}\Phi_{2\frac12} + 5Q_{\frac12}\Phi_{\frac12}q_1^4\\
&\quad - 30Q_{\frac12}\Phi_{\frac12}q_1 q_3 + 660Q_{\frac12}\Phi_{1\frac12}^2 - 10Q_{\frac12}\Phi_{1\frac12}q_1^3 - 30Q_{\frac12}\Phi_{1\frac12}q_3 + 60Q_{\frac12}\Phi_{3\frac12}q_1\\
&\quad - 120Q_{\frac12}\Phi_{4\frac12} + 12\Phi_5 - 120\Phi_3 Q_{\frac12}Q_{\frac32} - 60\Phi_3\Phi_{\frac12} - 120\Phi_2\Phi_{1\frac12}\\
&\quad - 20\Phi_1 Q_{\frac52}Q_{\frac32} - 30\Phi_1 Q_{\frac12}Q_{\frac72} + 360\Phi_1 Q_{\frac12}Q_{\frac32}\Phi_{\frac12} - 10\Phi_1 Q_{\frac12}Q_{\frac52}q_1\\
&\quad - 240\Phi_1\Phi_2 Q_{\frac12} - 60\Phi_1\Phi_{\frac12}q_1^2 + 60\Phi_1\Phi_{1\frac12}q_1 - 120\Phi_1\Phi_{2\frac12} + 360\Phi_0 Q_{\frac12}Q_{\frac32}\Phi_{1\frac12}\\
&\quad - 360\Phi_0\Phi_3 Q_{\frac12} + 120\Phi_0\Phi_2 Q_{\frac32} + 120\Phi_0\Phi_2 Q_{\frac12}q_1 + 60\Phi_0\Phi_1 Q_{\frac32}q_1\\
&\quad - 20\Phi_0\Phi_1 Q_{\frac52} + 720\Phi_0\Phi_1 Q_{\frac12}\Phi_{\frac12} - 180\Phi_0\Phi_1 Q_{\frac12}q_1^2 + 300\Phi_0\Phi_{\frac12}\Phi_{1\frac12}\\
&\quad - 90\Phi_0\Phi_{\frac12}q_1^3 + 90\Phi_0\Phi_{1\frac12}q_1^2 - 60\Phi_0\Phi_{3\frac12} + \theta(240\Phi_5 Q_{\frac12} + 40\Phi_3 Q_{\frac52}\\
&\quad - 1200\Phi_3 Q_{\frac12}\Phi_{\frac12} - 2400\Phi_2 Q_{\frac12}\Phi_{1\frac12} + 2\Phi_1 Q_{\frac92} - 120\Phi_1 Q_{\frac52}\Phi_{\frac12} + 2400\Phi_1 Q_{\frac12}\Phi_{\frac12}^2\\
&\quad - 2400\Phi_1 Q_{\frac12}\Phi_{2\frac12} - 600\Phi_1\Phi_3 + 240\Phi_1\Phi_2 q_1 - 120\Phi_0 Q_{\frac52}\Phi_{1\frac12} + 4800\Phi_0 Q_{\frac12}\Phi_{\frac12}\Phi_{1\frac12}\\
&\quad - 1200\Phi_0 Q_{\frac12}\Phi_{3\frac12} - 480\Phi_0\Phi_4 + 360\Phi_0\Phi_3 q_1 + 1920\Phi_0\Phi_2\Phi_{\frac12} - 120\Phi_0\Phi_2 q_1^2\\
&\quad - 720\Phi_0\Phi_1\Phi_{\frac12}q_1 + 1680\Phi_0\Phi_1\Phi_{1\frac12} + 20\Phi_0\Phi_1 q_1^3 - 120\Phi_0\Phi_1 q_3 + 660\Phi_{\frac12}^3 q_1\\
&\quad - 3540\Phi_{\frac12}^2\Phi_{1\frac12} - 40\Phi_{\frac12}^2 q_1^3 + 240\Phi_{\frac12}^2 q_3 + 360\Phi_{\frac12}\Phi_{1\frac12}q_1^2 - 960\Phi_{\frac12}\Phi_{2\frac12}q_1\\
&\quad + 1200\Phi_{\frac12}\Phi_{3\frac12} + \Phi_{\frac12}q_1^5 - 60\Phi_{\frac12}q_1^2 q_3 + 240\Phi_{\frac12}q_5 - 720\Phi_{1\frac12}^2 q_1 + 2400\Phi_{1\frac12}\Phi_{2\frac12}\\
&\quad - 5\Phi_{1\frac12}q_1^4 + 120\Phi_{1\frac12}q_1 q_3 + 20\Phi_{2\frac12}q_1^3 - 120\Phi_{2\frac12}q_3 - 60\Phi_{3\frac12}q_1^2 + 120\Phi_{4\frac12}q_1\\
&\quad - 120\Phi_{5\frac12}))/5.
\end{aligned}
$$

*The $Z_{k\frac12}$ series.*

$$
\begin{aligned}
Z_{\frac12} &= -2\theta\Phi_1 + \Phi_{\frac12},\\
Z_{\frac52} &= -2\Phi_1 Q_{\frac32} + \Phi_1 Q_{\frac12}q_1 + 2\Phi_0\Phi_1 - 4\Phi_{\frac12}^2 + \Phi_{\frac12}q_1^2 - 2\Phi_{1\frac12}q_1 + 2\Phi_{2\frac12}\\
&\quad + \theta(-4\Phi_3 + 12\Phi_1\Phi_{\frac12} + 12\Phi_0\Phi_{1\frac12}),\\
Z_{\frac92} &= -24\Phi_3 Q_{\frac32} + 24\Phi_3 Q_{\frac12}q_1 - 6\Phi_1 Q_{\frac72} + 72\Phi_1 Q_{\frac32}\Phi_{\frac12} + 2\Phi_1 Q_{\frac52}q_1\\
&\quad - 36\Phi_1 Q_{\frac12}\Phi_{\frac12}q_1 + 24\Phi_1 Q_{\frac12}\Phi_{1\frac12} - 36\Phi_1 Q_{\frac12}q_3 + 48\Phi_1\Phi_2 + 72\Phi_0 Q_{\frac32}\Phi_{1\frac12}\\
&\quad - 72\Phi_0 Q_{\frac12}\Phi_{1\frac12}q_1 + 72\Phi_0\Phi_3 - 48\Phi_0\Phi_2 q_1 - 144\Phi_0\Phi_1\Phi_{\frac12} + 48\Phi_0\Phi_1 q_1^2\\
&\quad + \theta(-48\Phi_5 + 240\Phi_3\Phi_{\frac12} + 480\Phi_2\Phi_{1\frac12} - 480\Phi_1\Phi_{\frac12}^2 + 480\Phi_1\Phi_{2\frac12}\\
&\quad - 960\Phi_0\Phi_{\frac12}\Phi_{1\frac12} + 240\Phi_0\Phi_{3\frac12}) + 132\Phi_{\frac12}^3 - 24\Phi_{\frac12}^2 q_1^2 + 144\Phi_{\frac12}\Phi_{1\frac12}q_1\\
&\quad - 192\Phi_{\frac12}\Phi_{2\frac12} + \Phi_{\frac12}q_1^4 - 24\Phi_{\frac12}q_1 q_3 - 144\Phi_{1\frac12}^2 - 4\Phi_{1\frac12}q_1^3 + 24\Phi_{1\frac12}q_3\\
&\quad + 12\Phi_{2\frac12}q_1^2 - 24\Phi_{3\frac12}q_1 + 24\Phi_{4\frac12}.
\end{aligned}
$$

*Gradings.* There are two points of view on symmetries: as on functions and as on vector fields $\mathcal{E}_f$ (see Subsection 1.1). For functions we have:

$$\begin{array}{llll} [Y_1] = 5/2, & [Y_3] = 9/2, & [Y_5] = 13/2, & \text{odd;} \\ [Y_{\frac{3}{2}}] = 3, & [Y_{\frac{7}{2}}] = 5, & [Y_{\frac{11}{2}}] = 7, & \text{even;} \\ [Z_1] = 5/2, & [Z_3] = 7/2, & [Z_5] = 13/2, & \text{odd;} \\ [Z_{\frac{1}{2}}] = 2, & [Z_{\frac{5}{2}}] = 4, & [Z_{\frac{9}{2}}] = 6, & \text{even.} \end{array}$$

For vector fields we have:

$$\begin{array}{llll} [\mathcal{E}_{Y_1}] = 1, & [\mathcal{E}_{Y_3}] = 3, & [\mathcal{E}_{Y_5}] = 5, & \text{even;} \\ [\mathcal{E}_{Y_{\frac{3}{2}}}] = 3/2, & [\mathcal{E}_{Y_{\frac{7}{2}}}] = 7/2, & [\mathcal{E}_{Y_{\frac{11}{2}}}] = 11/2, & \text{odd;} \\ [\mathcal{E}_{Z_1}] = 1, & [\mathcal{E}_{Z_3}] = 3, & [\mathcal{E}_{Z_5}] = 5, & \text{even;} \\ [\mathcal{E}_{Z_{\frac{1}{2}}}] = 1/2, & [\mathcal{E}_{Z_{\frac{5}{2}}}] = 5/2, & [\mathcal{E}_{Z_{\frac{9}{2}}}] = 9/2, & \text{odd.} \end{array}$$

Note also that the symmetries $Y_\alpha$ do not depend on $\theta$, while $Z_\alpha$ are linear functions with respect to $\theta$.

2.3. **Seed generating functions.** Solving equation (5), which in our case is of the form

$$\widetilde{D}_t(f) = -\widetilde{D}_\theta^6(f) + 6\Phi_{\frac{1}{2}}\widetilde{D}_\theta^2(f) - 3\Phi_0\widetilde{D}_\theta^3(f),$$

we found a number of solutions that serve as seed generating functions for constructing infinite hierarchies and used to construct *nonlocal forms* (see Subsection 2.5 below). These generating functions are:

*The $F_k$ series.*

$$\begin{aligned} F_0 &= 1, \\ F_2 &= \Phi_{\frac{1}{2}}, \\ F_4 &= (-2\Phi_0\Phi_1 + 3\Phi_{\frac{1}{2}}^2 - \Phi_{2\frac{1}{2}})/3. \end{aligned}$$

*The $F_{k\frac{1}{2}}$ series.*

$$\begin{aligned} F_{\frac{1}{2}} &= Q_{\frac{1}{2}}, \\ F_{\frac{5}{2}} &= (Q_{\frac{5}{2}} - 12Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 6\Phi_1 + 6\Phi_0 q_1)/6, \\ F_{\frac{9}{2}} &= (Q_{\frac{9}{2}} - 40Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} + 720Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 - 240Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} + 120\Phi_3 + 120\Phi_2 q_1 \\ &\quad - 480\Phi_1\Phi_{\frac{1}{2}} + 60\Phi_1 q_1^2 - 480\Phi_0\Phi_1 Q_{\frac{1}{2}} - 420\Phi_0\Phi_{\frac{1}{2}}q_1 - 240\Phi_0\Phi_{1\frac{1}{2}} + 20\Phi_0 q_1^3 \\ &\quad - 120\Phi_0 q_3)/20. \end{aligned}$$

*The $G_k$ series.*

$$\begin{aligned} G_0 &= \theta Q_{\frac{1}{2}}, \\ G_2 &= (3Q_{\frac{1}{2}}Q_{\frac{3}{2}} + 6\Phi_0 Q_{\frac{1}{2}} + \theta Q_{\frac{5}{2}} - 12\theta Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 6\theta\Phi_1 + 6\theta\Phi_0 q_1)/3, \\ G_4 &= (-10Q_{\frac{5}{2}}Q_{\frac{3}{2}} + 15Q_{\frac{1}{2}}Q_{\frac{7}{2}} + 120Q_{\frac{1}{2}}Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} - 5Q_{\frac{1}{2}}Q_{\frac{5}{2}}q_1 - 120\Phi_2 Q_{\frac{1}{2}} \\ &\quad - 60\Phi_1 Q_{\frac{3}{2}} - 60\Phi_0 Q_{\frac{3}{2}}q_1 - 20\Phi_0 Q_{\frac{5}{2}} + 420\Phi_0 Q_{\frac{1}{2}}\Phi_{\frac{1}{2}} + 90\Phi_0 Q_{\frac{1}{2}}q_1^2 \\ &\quad - 120\Phi_0\Phi_1 - \theta Q_{\frac{9}{2}} + 40\theta Q_{\frac{5}{2}}\Phi_{\frac{1}{2}} - 720\theta Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}^2 + 240\theta Q_{\frac{1}{2}}\Phi_{2\frac{1}{2}} - 120\theta\Phi_3 \\ &\quad - 120\theta\Phi_2 q_1 + 480\theta\Phi_1\Phi_{\frac{1}{2}} - 60\theta\Phi_1 q_1^2 + 480\theta\Phi_0\Phi_1 Q_{\frac{1}{2}} + 420\theta\Phi_0\Phi_{\frac{1}{2}}q_1 \\ &\quad + 240\theta\Phi_0\Phi_{1\frac{1}{2}} - 20\theta\Phi_0 q_1^3 + 120\theta\Phi_0 q_3)/90. \end{aligned}$$

*The $G_{k\frac{1}{2}}$ series.*

$$
\begin{aligned}
G_{-\frac{1}{2}} &= \theta,\\
G_{\frac{3}{2}} &= -Q_{\frac{3}{2}} + Q_{\frac{1}{2}}q_1 + 2\Phi_0 - 4\theta\Phi_{\frac{1}{2}},\\
G_{\frac{7}{2}} &= (3Q_{\frac{7}{2}} - 24Q_{\frac{3}{2}}\Phi_{\frac{1}{2}} - Q_{\frac{5}{2}}q_1 + 6Q_{\frac{1}{2}}\Phi_{\frac{1}{2}}q_1 - 12Q_{\frac{1}{2}}\Phi_{1\frac{1}{2}} + 18Q_{\frac{1}{2}}q_3 - 24\Phi_2\\
&\quad - 12\Phi_1 q_1 + 84\Phi_0\Phi_{\frac{1}{2}} + 6\Phi_0 q_1^2 + 96\theta\Phi_0\Phi_1 - 144\theta\Phi_{\frac{1}{2}}^2 + 48\theta\Phi_{2\frac{1}{2}})/6.
\end{aligned}
$$

*Gradings.* These generating functions have the following gradings and parities:

$$
\begin{array}{llll}
[F_0] = 0, & [F_2] = 2, & [F_4] = 4, & \text{even};\\
[F_{\frac{1}{2}}] = 1/2, & [F_{\frac{5}{2}}] = 5/2, & [F_{\frac{9}{2}}] = 9/2, & \text{odd};\\
[G_0] = 0, & [G_2] = 2, & [G_4] = 4, & \text{even};\\
[G_{-\frac{1}{2}}] = -1/2, & [G_{\frac{3}{2}}] = 3/2, & [G_{\frac{7}{2}}] = 7/2, & \text{odd}.
\end{array}
$$

Note again that the generating functions $F_\alpha$ do not depend on $\theta$, while $G_\alpha$ are linear functions with respect to $\theta$.

2.4. **Nonlocal vectors.** We consider now to the $\ell^*$-extension of equation (2). The additional coordinates on this extension are denoted by $P = P_0$, $P_{\frac{1}{2}}$, $P_1$, etc.

Now we introduce nonlocal variables in the $\ell^*$-extension that we call *nonlocal vectors* and which are defined by

$$
\begin{array}{lll}
(P_{Y_1})_\theta = Y_1 P_0, & (P_{Y_3})_\theta = Y_3 P_0, & (P_{Y_5})_\theta = Y_5 P_0;\\
(P_{Y_{\frac{3}{2}}})_\theta = Y_{\frac{3}{2}} P_0, & (P_{Y_{\frac{7}{2}}})_\theta = Y_{\frac{7}{2}} P_0, & (P_{Y_{\frac{11}{2}}})_\theta = Y_{\frac{11}{2}} P_0;\\
(P_{Z_1})_\theta = Z_1 P_0, & (P_{Z_3})_\theta = Z_3 P_0, & (P_{Z_5})_\theta = Z_5 P_0;\\
(P_{Z_{\frac{1}{2}}})_\theta = Z_{\frac{1}{2}} P_0, & (P_{Z_{\frac{5}{2}}})_\theta = Z_{\frac{5}{2}} P_0, & (P_{Z_{\frac{9}{2}}})_\theta = Z_{\frac{9}{2}} P_0,
\end{array}
$$

where the symmetries $Y_\alpha$ and $Z_\alpha$ were described in Subsection 2.2.

The $x$- and $t$-components of these variables are given in [6].

*Gradings.* The variable $P_0$ is even and we assign grading 0 to it. Then $P_k$ are also even variables with $[P_k] = k$ while $P_{k\frac{1}{2}}$ are odd and $[P_{k\frac{1}{2}}] = (2k+1)/2$. Consequently,

$$
\begin{array}{llll}
[P_{Y_1}] = 2, & [P_{Y_3}] = 4, & [P_{Y_5}] = 6, & \text{even};\\
[P_{Y_{\frac{3}{2}}}] = 5/2, & [P_{Y_{\frac{7}{2}}}] = 9/2, & [P_{Y_{\frac{11}{2}}}] = 13/2, & \text{odd};\\
[P_{Z_1}] = 2, & [P_{Z_3}] = 4, & [P_{Z_5}] = 6, & \text{even};\\
[P_{Z_{\frac{1}{2}}}] = 3/2, & [P_{Z_{\frac{5}{2}}}] = 7/2, & [P_{Z_{\frac{9}{2}}}] = 11/2, & \text{odd}.
\end{array}
$$

2.5. **Nonlocal forms.** Passing to the $\ell$-extension of equation (2), we introduce the additional coordinates on this extension that are denoted by $\Omega = \Omega_0$, $\Omega_{\frac{1}{2}}$, $\Omega_1$, etc.

Now we introduce nonlocal variables in the $\ell$-extension called *nonlocal forms* and described by

$$
\begin{array}{lll}
(\Omega_{F_0})_\theta = \Omega_0 F_0, & (\Omega_{F_2})_\theta = \Omega_0 F_2, & (\Omega_{F_4})_\theta = \Omega_0 F_4;\\
(\Omega_{F_{\frac{1}{2}}})_\theta = \Omega_0 F_{\frac{1}{2}}, & (\Omega_{F_{\frac{5}{2}}})_\theta = \Omega_0 F_{\frac{5}{2}}, & (\Omega_{F_{\frac{9}{2}}})_\theta = \Omega_0 F_{\frac{9}{2}};\\
(\Omega_{G_0})_\theta = \Omega_0 G_0, & (\Omega_{G_2})_\theta = \Omega_0 G_2, & (\Omega_{G_4})_\theta = \Omega_0 G_4;\\
(\Omega_{G_{-\frac{1}{2}}})_\theta = \Omega_0 G_{-\frac{1}{2}}, & (\Omega_{G_{\frac{3}{2}}})_\theta = \Omega_0 G_{\frac{3}{2}}, & (\Omega_{G_{\frac{7}{2}}})_\theta = \Omega_0 G_{\frac{7}{2}},
\end{array}
$$

where the generating functions $F_\alpha$ and $G_\alpha$ were described in Subsection 2.3.

The $x$- and $t$-components of these variables are given in [6].

*Gradings.* The variable $\Omega_0$ is even and we assign grading 0 to it. Then $\Omega_k$ are also even variables with $[\Omega_k] = k$, while $\Omega_{k\frac{1}{2}}$ are odd and $[\Omega_{k\frac{1}{2}}] = (2k+1)/2$. Consequently,

$$\begin{array}{llll}
[\Omega_{F_0}] = -1/2, & [\Omega_{F_2}] = 3/2, & [\Omega_{F_4}] = 7/2, & \text{odd;} \\
[\Omega_{F_{\frac{1}{2}}}] = 0, & [\Omega_{F_{\frac{5}{2}}}] = 2, & [\Omega_{F_{\frac{9}{2}}}] = 4, & \text{even;} \\
[\Omega_{G_0}] = -1/2, & [\Omega_{G_2}] = 3/2, & [\Omega_{G_4}] = 7/2, & \text{odd;} \\
[\Omega_{G_{-\frac{1}{2}}}] = -1, & [\Omega_{G_{\frac{3}{2}}}] = 1, & [\Omega_{G_{\frac{7}{2}}}] = 3, & \text{even.}
\end{array}$$

2.6. **Recursion operators for symmetries.** Using the method described in Subsection 1.5, we found two nontrivial solutions of the linearized equation in the $\ell$-extension enriched with nonlocal variables. The first one is

$$\begin{aligned}
R_1 = &-Q_{\frac{1}{2}}\Omega_{F_0}\Phi_{\frac{1}{2}} - 2\Phi_1\Omega_{G_0} - \Phi_1\Omega_{F_0} + 2\Phi_1 Q_{\frac{1}{2}}\Omega_{G_{-\frac{1}{2}}} \\
&- 2\Phi_0\Omega_{\frac{1}{2}} + \theta\Omega_{F_0}\Phi_{\frac{1}{2}}q_1 - \theta\Omega_{F_0}\Phi_{1\frac{1}{2}} + 2\theta\Phi_1 Q_{\frac{1}{2}}\Omega_{F_0} \\
&+ 2\theta\Phi_1\Omega_{F_{\frac{1}{2}}} - \Omega_{F_{\frac{1}{2}}}\Phi_{\frac{1}{2}} + \Omega_{G_{-\frac{1}{2}}}\Phi_{\frac{1}{2}}q_1 - \Omega_{G_{-\frac{1}{2}}}\Phi_{1\frac{1}{2}} \\
&- 2\Omega_0\Phi_{\frac{1}{2}} + \Omega_2.
\end{aligned}$$

The operator corresponding to the first solution is

$$\begin{aligned}
\Delta_{R_1} = &D_\theta^4 - 2\Phi_0 D_\theta - 2\Phi_{\frac{1}{2}} \\
&- (Y_1 + Z_1)D_\theta^{-1} \circ F_0 - Z_{\frac{1}{2}}D_\theta^{-1} \circ F_{\frac{1}{2}} - Y_{\frac{3}{2}}D_\theta^{-1} \circ G_{-\frac{1}{2}} - 2Y_1 D_\theta^{-1} \circ G_0.
\end{aligned}$$

The second solution is given in [6] and corresponds to the operator $\Delta_{R_1}^2$.

*Gradings.* The operator $R_1$ is even and its grading is 2.

2.7. **Recursion operators for generating functions.** Using the method described in Subsection 1.6, we found three nontrivial solutions of the adjoint linearized equation in the $\ell^*$-extension enriched with nonlocal variables. The first one is

$$L_1 = Q_{\frac{1}{2}}P_{Z_{\frac{1}{2}}} + 2\Phi_0 P_{\frac{1}{2}} + \theta P_{Y_{\frac{3}{2}}} + 2\theta Q_{\frac{1}{2}}P_{Y_1} - 4\Phi_{\frac{1}{2}}P_0 + P_{Y_1} + P_{Z_1} + P_2.$$

The operator corresponding to the first solution is

$$\begin{aligned}
\Delta_{L_1} = &D_\theta^4 + 2\Phi_0 D_\theta - 4\Phi_{\frac{1}{2}} \\
&+ (F_0 + 2G_0)D_\theta^{-1} \circ Y_1 + G_{-\frac{1}{2}}D_\theta^{-1} \circ Y_{\frac{3}{2}} + F_0 D_\theta^{-1} \circ Z_1 + F_{\frac{1}{2}}D_\theta^{-1} \circ Z_{\frac{1}{2}}.
\end{aligned}$$

The second and third solutions are given in [6] and correspond to the operators $\Delta_{L_1}^2$ and $\Delta_{L_1}^3$, resp.

*Gradings.* The operator $L_1$ is even and its grading is 2.

2.8. **Hamiltonian structures.** Using the method described in Subsection 1.7, we found three nontrivial solutions of the linearized equation in the $\ell^*$-extension enriched with nonlocal variables. The first one is

$$K_1 = P_{2\frac{1}{2}} - P_{\frac{1}{2}}\Phi_{\frac{1}{2}} - 2\Phi_1 P_0 - 3\Phi_0 P_1.$$

The operator corresponding to the first solution is

$$\Delta_{K_1} = D_\theta^5 - 3\Phi_0 D_\theta^2 - \Phi_{\frac{1}{2}}D_\theta - 2\Phi_1.$$

This operator satisfies criteria (15) and (16) and thus is Hamiltonian. Moreover, there exists a conservation law (corresponding to the nonlocal variable $q_3$)

$$
\begin{aligned}
&X = \Phi_0\Phi_{\frac12},\\
&T = -2\Phi_1\Phi_2 + \Phi_0\Phi_3 - 9\Phi_0\Phi_1\Phi_{\frac12} + 4\Phi_{\frac12}^3 - 2\Phi_{\frac12}\Phi_{2\frac12} + \Phi_{1\frac12}^2
\end{aligned}
$$

such that our equation can be represented as

$$\Phi_t = \Delta_{K_1}\frac{\delta}{\delta\Phi}\Big(-\frac12 X\Big),$$

and so (18) is also satisfied.

The second Hamiltonian structure is of the form

$$
\begin{aligned}
K_2 = {}& -P_{Z_{\frac12}}\Phi_{\frac12}q_1 + P_{Z_{\frac12}}\Phi_{1\frac12} - P_{Y_{\frac32}}\Phi_{\frac12} + P_{4\frac12} - 3P_{2\frac12}\Phi_{\frac12} - 3P_{1\frac12}\Phi_{1\frac12}\\
&+ 3P_{\frac12}\Phi_{\frac12}^2 - P_{\frac12}\Phi_{2\frac12} - 2Q_{\frac12}\Phi_{\frac12}P_{Y_1} - 2\Phi_3P_0 - 7\Phi_2P_1 - 2\Phi_1Q_{\frac12}P_{Z_{\frac12}}\\
&+ 9\Phi_1\Phi_{\frac12}P_0 - 2\Phi_1P_{Z_1} - 9\Phi_1P_2 - \Phi_0\Phi_1P_{\frac12} + 13\Phi_0\Phi_{\frac12}P_1 + 7\Phi_0\Phi_{1\frac12}P_0\\
&- 5\Phi_0P_3 + 2\theta\Phi_1P_{Y_{\frac32}} + 4\theta\Phi_1Q_{\frac12}P_{Y_1} + 2\theta\Phi_{\frac12}q_1P_{Y_1} - 2\theta\Phi_{1\frac12}P_{Y_1}.
\end{aligned}
$$

The corresponding operator is

$$
\begin{aligned}
\Delta_{K_2} = {}& D_\theta^9 - 5\Phi_0D_\theta^6 - 3\Phi_{\frac12}D_\theta^5 - 9\Phi_1D_\theta^4 - 3\Phi_{1\frac12}D_\theta^3 + (13\Phi_0\Phi_{\frac12} - 7\Phi_2)D_\theta^2\\
&+ (3\Phi_{\frac12}^2 - \Phi_{2\frac12} - \Phi_0\Phi_1)D_\theta + (9\Phi_1\Phi_{\frac12} + 7\Phi_0\Phi_{1\frac12} - 2\Phi_3)\\
&+ Y_{\frac32}D_\theta^{-1}\circ Z_{\frac12} - Z_{\frac12}D_\theta^{-1}\circ Y_{\frac32} - 2Y_1D_\theta^{-1}\circ Z_1 - 2Z_1D_\theta^{-1}\circ Y_1.
\end{aligned}
$$

The third solution is given in [6] (see also Remark 9 below).

*Gradings.* The operator $\Delta_{K_1}$ is odd and of grading $5/2$. The operator $\Delta_{K_2}$ is also odd and of grading $9/2$.

2.9. **Symplectic structures.** Using the method described in Subsection 1.8, we found three nontrivial solutions of the adjoint linearized equation in the $\ell$-extension enriched with nonlocal variables. The first one is

$$J_1 = \Omega_{G_0} + \Omega_{F_0} - Q_{\frac12}\Omega_{G_{-\frac12}} + \theta Q_{\frac12}\Omega_{F_0} + \theta\Omega_{F_{\frac12}}.$$

The operator corresponding to the first solution is

$$\Delta_{J_1} = (F_0 + G_0)D_\theta^{-1}\circ F_0 + G_{-\frac12}D_\theta^{-1}\circ F_{\frac12} - F_{\frac12}D_\theta^{-1}\circ G_{-\frac12} + F_0D_\theta^{-1}\circ G_0.$$

This operator satisfies criteria (19) and (20) and thus is symplectic.

The second solution is of the form

$$
\begin{aligned}
J_2 = {}& (3\Omega_{G_2} - 12\Omega_{G_0}\Phi_{\frac12} - 12\Omega_{F_2} - 12\Omega_{F_0}\Phi_{\frac12} + 6\Omega_{1\frac12} - 3Q_{\frac32}\Omega_{F_{\frac12}}\\
&- Q_{\frac52}\Omega_{G_{-\frac12}} + 3Q_{\frac12}Q_{\frac32}\Omega_{F_0} + 3Q_{\frac12}\Omega_{F_{\frac12}}q_1 + 12Q_{\frac12}\Omega_{G_{-\frac12}}\Phi_{\frac12}\\
&- 3Q_{\frac12}\Omega_{G_{\frac32}} - 6\Phi_1\Omega_{G_{-\frac12}} + 6\Phi_0Q_{\frac12}\Omega_{F_0} + 6\Phi_0\Omega_{F_{\frac12}} - 6\Phi_0\Omega_{G_{-\frac12}}q_1\\
&+ 6\Phi_0\Omega_0 + \theta Q_{\frac52}\Omega_{F_0} - 12\theta Q_{\frac12}\Omega_{F_2} - 12\theta Q_{\frac12}\Omega_{F_0}\Phi_{\frac12} + 6\theta\Phi_1\Omega_{F_0}\\
&+ 6\theta\Phi_0\Omega_{F_0}q_1 - 12\theta\Omega_{F_{\frac12}}\Phi_{\frac12} + 6\theta\Omega_{F_{\frac52}})/6.
\end{aligned}
$$

The corresponding operator is

$$
\begin{aligned}
\Delta_{J_2} = {}& D_\theta^3 + \Phi_0 + (\frac12 G_2 - 2F_2)D_\theta^{-1}\circ F_0\\
&- 2(F_0 + G_0)D_\theta^{-1}\circ F_2 + \frac12 G_{\frac32}D_\theta^{-1}\circ F_{\frac12} + G_{-\frac12}D_\theta^{-1}\circ F_{\frac52}\\
&- 2F_2D_\theta^{-1}\circ G_0 + \frac12 F_0D_\theta^{-1}\circ G_2 - 6F_{\frac52}D_\theta^{-1}\circ G_{-\frac12} - \frac12 F_{\frac12}D_\theta^{-1}\circ G_{\frac32}.
\end{aligned}
$$

The third solution is given in [6] (see Remark 9).

*Gradings.* The operator $\Delta_{J_1}$ is odd and of grading $-1/2$. The second operator is also odd and its grading equals $3/2$.

2.10. **Interrelations.** Using the symmetries computed in Subsection 2.2 and applying the recursion operator obtained in Subsection 2.6, we get four infinite series of (generally, nonlocal) symmetries

$$
\begin{array}{lll}
Y_{2k-1}, & [Y_{2k-1}] = (4k+1)/2, & \text{odd}, \\
Y_{\frac{4k-1}{2}}, & [Y_{\frac{4k-1}{2}}] = 2k+1, & \text{even}, \\
Z_{2k-1}, & [Z_{2k-1}] = (4k+1)/2, & \text{odd}, \\
Z_{\frac{4k-3}{2}}, & [Z_{\frac{4k-3}{2}}] = 2k, & \text{even},
\end{array}
$$

$k = 1, 2, \dots$

In a similar war, using the results of Subsections 2.3 and 2.7, we get four infinite series of generating functions

$$
\begin{array}{lll}
F_{2k-2}, & [F_{2k-2}] = 2k-2, & \text{even}, \\
F_{\frac{4k-3}{2}}, & [F_{\frac{4k-3}{2}}] = (4k-3)/2, & \text{odd}, \\
G_{2k}, & [G_{2k}] = 2k, & \text{even}, \\
G_{\frac{4k-5}{2}}, & [G_{\frac{4k-5}{2}}] = (4k-5)/2, & \text{odd},
\end{array}
$$

$k = 1, 2, \dots$

These series are related to each other (up to rational coefficients) by the operators of Subsections 2.6–2.9 in the following way:

$$
\begin{tikzcd}
& Y_{2k-1} \arrow[r, "\Delta_{R_1}"] \arrow[d, "\Delta_{J_1}"] & Y_{2k+1} \arrow[d, "\Delta_{J_1}"] \\
F_{2k-2} \arrow[ur, "\Delta_{K_1}"] \arrow[r, "\Delta_{L_1}"'] & F_{2k} \arrow[ur, "\Delta_{K_1}"'] \arrow[r, "\Delta_{L_1}"'] & F_{2k+2}
\end{tikzcd}
\qquad
\begin{tikzcd}
& Z_{2k-1} \arrow[r, "\Delta_{R_1}"] \arrow[d, "\Delta_{J_1}"] & Z_{2k+1} \arrow[d, "\Delta_{J_1}"] \\
G_{2k} \arrow[ur, "\Delta_{K_1}"] \arrow[r, "\Delta_{L_1}"'] & G_{2k+2} \arrow[ur, "\Delta_{K_1}"'] \arrow[r, "\Delta_{L_1}"'] & G_{2k+4}
\end{tikzcd}
$$

$$
\begin{tikzcd}
& Y_{\frac{4k-1}{2}} \arrow[r, "\Delta_{R_1}"] \arrow[d, "\Delta_{J_1}"] & Y_{\frac{4k+3}{2}} \arrow[d, "\Delta_{J_1}"] \\
F_{\frac{4k-3}{2}} \arrow[ur, "\Delta_{K_1}"] \arrow[r, "\Delta_{L_1}"'] & F_{\frac{4k+1}{2}} \arrow[ur, "\Delta_{K_1}"'] \arrow[r, "\Delta_{L_1}"'] & F_{\frac{4k+5}{2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
& Z_{\frac{4k-3}{2}} \arrow[r, "\Delta_{R_1}"] \arrow[d, "\Delta_{J_1}"] & Z_{\frac{4k+1}{2}} \arrow[d, "\Delta_{J_1}"] \\
G_{\frac{4k-5}{2}} \arrow[ur, "\Delta_{K_1}"] \arrow[r, "\Delta_{L_1}"'] & G_{\frac{4k-1}{2}} \arrow[ur, "\Delta_{K_1}"'] \arrow[r, "\Delta_{L_1}"'] & G_{\frac{4k+3}{2}}
\end{tikzcd}
$$

*Remark* 8. Actually, there exists another hierarchy of symmetries $S_{2k}$, $k = 0, 1, \dots$, with the seed element

$$S_0 = 6(-\Phi_3 + 3\Phi_1\Phi_{\frac{1}{2}} + 3\Phi\Phi_{1\frac{1}{2}})t + 2\Phi_1 x + \theta\Phi_{\frac{1}{2}} + 3\Phi$$

(the scaling symmetry). All these symmetries are odd, linear with respect to $x$, $t$, and $\theta$, and have grading $[S_{2k}] = (4k+3)/2$, see [16].

*Remark* 9 (cf. [1]). Let us clarify the relations between the structures described above. First, it should be noted that the Hamiltonian structures $K_1$ and $K_2$ are *compatible*, i.e., their Schouten bracket vanishes (or, their linear combination is a Hamiltonian structure again). More over, they are related to each other by the rerecursion operator $R_1$: $\Delta_{K_2} = \Delta_{R_1} \circ \Delta_{K_1}$. Consequently, an infinite series of (nonlocal) compatible Hamiltonian structures $K_i$ arises, such that $\Delta_{K_{i+1}} = \Delta_{R_1} \circ \Delta_{K_i}$. In a similar way, we have an infinite series of symplectic structures related by the operator $L_1$. The inverse of each Hamiltonian structure, if it makes sense, is a symplectic structure and vice versa.

Second, in an obvious way all natural powers of recursion operators are also recursion operators. If $R$ is a recursion operator for symmetries, the adjoint $R^*$ is a recursion for generating functions and vice versa.

## 3. Conclusion

The study of the $N = 1$ supersymmetric KdV equation exposed in this paper demonstrates the power and efficiency of the geometrical methods elaborated in [2] and [7]. In particular, we found recursion operators for symmetries and generating functions, Hamiltonian and symplectic structures, constructed five infinite series of symmetries. The research was based on new geometrical methods giving rise to efficient computational algorithms.

Our experience shows that the methods applied are of a universal nature and may be used to analyze a lot of other equations, both classical and supersymmetric. In particular, from technical point of view, the canonical representation of nonlocal operators (see Subsection 1.9) seems to be quite efficient and convenient when dealing with such operators. Note that all nonlocal operators constructed in this paper are represented in the canonical form.

We strongly believe that the majority of the problems formulated in [15] can be solved by our methods. We plan to demonstrate this in forthcoming publications. Note in particular that the nonlocal Hamiltonian structure indicated in [15] is inverse to our symplectic structure $J_1$.

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